Solutions practical 1

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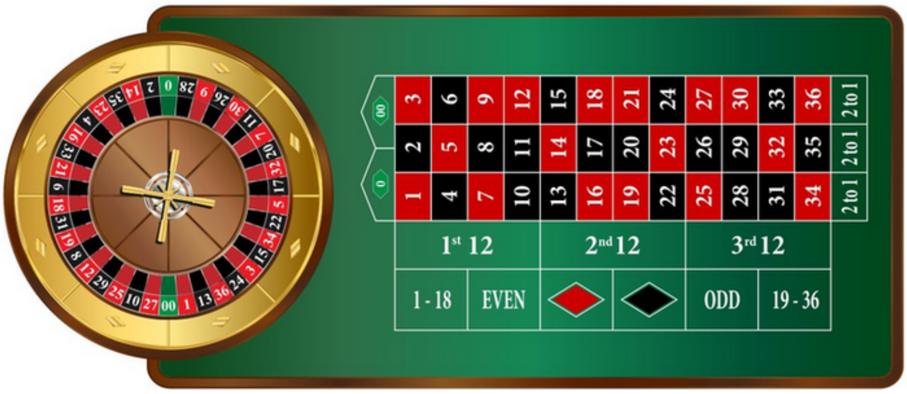
SEM 2: Structural Equation Modeling

The *American Roulette* casino game is slightly different from the regular (French) roulette game described in the lecture, in that an extra "00" field is added, leading to a total of 38 spaces the ball can fall on:



The following table gives an overview of some bets you can place:

bet type	payout	probability	expected value	standard deviation
single number (e.g., number 17)	35 to 1	1/38	...	...
single row (e.g., 4, 5, and 6)	11 to 1	...	...	...
single column (e.g., all values in first column, except for 0)	2 to 1	...	...	...
all red or black numbers	1 to 1	...	...	...
top line bet (bet on numbers 00, 0, 1, 2, and 3)	6 to 1	...	...	...

The payout of 35 to 1 indicates that betting € 10 on the right number leads to a profit of € 350, and betting € 10 on the wrong number leads to losing the 10 Euro.

Exercise 1 Fill in the missing cells in the table above, assuming a bet of € 10. ■

Solution:

bet type	payout	probability	expected value	standard deviation
single number (e.g., number 17)	35 to 1	1/38	-€ 0.53	57.62
single row (e.g., 4, 5, and 6)	11 to 1	3/38	-€ 0.53	32.36
single column (e.g., all values in first column, except for 0)	2 to 1	12/38	-€ 0.53	13.94
all red or black numbers	1 to 1	18/38	-€ 0.53	9.99
top line bet (bet on numbers 00, 0, 1, 2, and 3)	6 to 1	5/38	-€ 0.79	23.66

In Dungeons & Dragons (DnD; 3rd edition), I play a half-orc Barbarian that is not very smart, but can use the largest weapons possible. Three such weapons would be the following:

(a) A magical greatsword with 2d6 + 1 damage

(b) A magical flail with 1d10 + 2 damage

(c) A magical greataxe with 1d12 + 1 damage

In DnD, the notation "XdY + Z" stands for "throw X dices with Y sides, and add Z". This makes use of some special dices that have more sides (e.g., a 10-sided and a 12-sided die). The final value is the damage a weapon does.

Exercise 2 Calculate the expected value and standard deviation of each of these weapons. Which should I prefer? ■

Solution:
The expected value of a 6-sided dice is:

```
evD6 <- sum((1/6) * (1:6))
```

Which makes the expected value of the magical greatsword:

```
2 * evD6 + 1

## [1] 8
```

and its standard deviation:

```
var <- 2 * sum((1/6) * (1:6 - evD6)^2) # Note, we throw 2 indepenent dice
sqrt(var)

## [1] 2.415229
```

Similarly, for the flail:

```
# Expected value of the 1d10:
evD10 <- sum((1/10) * (1:10))
# Expected damage:
evD10 + 2

## [1] 7.5

# Standard deviation:
var <-  sum((1/10) * (1:10 - evD10)^2)
sqrt(var)

## [1] 2.872281
```

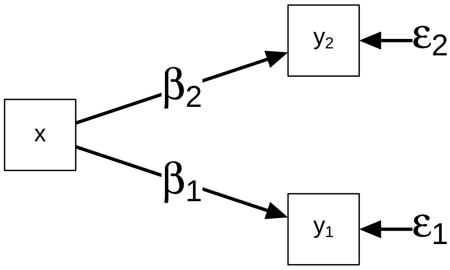
and for the axe:

```r
# Expected value of the 1d12:
evD12 <- sum((1/12) * (1:12))
# Expected damage:
evD12 + 1

## [1] 7.5

# Standard deviation:
var <-  sum((1/12) * (1:12 - evD12)^2)
sqrt(var)

## [1] 3.452053
```

The greatsword is clearly superior in terms of average damage (high expected value) as well as reliability (low standard deviation). The axe could also be prefered because it has a better change or doing a lot of damage in one hit though due to its higher standard deviation (and is way cooler than a sword). But it also has a high chance of doing a little damage in one hit...

The following path diagram (ignoring the variance of the exogenous variable x):

```
## Registered S3 methods overwritten by 'huge':
##   method    from
##   plot.sim  BDgraph
##   print.sim BDgraph
```



encodes the following structural equations (ignoring intercepts):

$$
\begin{aligned}
y_{i1} &= \beta_1 x_i + \varepsilon_{i1} \\
y_{i2} &= \beta_2 x_i + \varepsilon_{i2}
\end{aligned}
$$

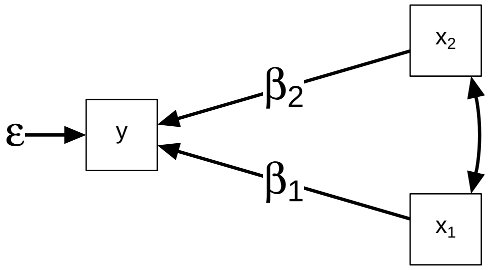
Exercise 3
Derive $\text{Var}(y_1)$ and $\text{Cov}(y_1, y_2)$.

Solution:

$$
\begin{aligned}
\text{Var}(y_1) &= \text{Var}(\beta_1 x + \varepsilon_1) \\
&= \text{Cov}(\beta_1 x + \varepsilon_1, \beta_1 x + \varepsilon_1) \\
&= \text{Cov}(\beta_1 x, \beta_1 x + \varepsilon_1) + \text{Cov}(\varepsilon_1, \beta_1 x + \varepsilon_1) \\
&= \text{Cov}(\beta_1 x, \beta_1 x) + \text{Cov}(\beta_1 x, \varepsilon_1) \\
&+ \text{Cov}(\varepsilon_1, \beta_1 x) + \text{Cov}(\varepsilon_1, \varepsilon_1) \\
&= \beta_1^2 \text{Cov}(x, x) + 0 + 0 + \text{Cov}(\varepsilon_1, \varepsilon_1) \\
&= \beta_1^2 \text{Var}(x) + \text{Var}(\varepsilon_1)
\end{aligned}
$$

$$
\begin{aligned}
\text{Cov}(y_1, y_2) &= \text{Cov}(\beta_1 x + \varepsilon_1, \beta_2 x + \varepsilon_2) \\
&= \text{Cov}(\beta_1 x, \beta_2 x + \varepsilon_2) + \text{Cov}(\varepsilon_1, \beta_2 x + \varepsilon_2) \\
&= \text{Cov}(\beta_1 x, \beta_2 x) + \text{Cov}(\beta_1 x, \varepsilon_2) \\
&+ \text{Cov}(\varepsilon_1, \beta_2 x) + \text{Cov}(\varepsilon_1, \varepsilon_2) \\
&= \beta_1 \beta_2 \text{Cov}(x, x) + 0 + 0 + 0 \\
&= \beta_1 \beta_2 \text{Var}(x)
\end{aligned}
$$

Now consider the following path diagram:

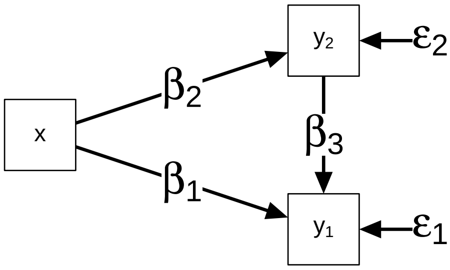


Exercise 4
Write down the structural equation for y and derive $\text{Var}(y)$.

Solution:

$$
\begin{aligned}
\text{Var}(y) &= \text{Var}(\beta_1 x_1 + \beta_2 x_2 + \varepsilon) \\
&= \text{Cov}(\beta_1 x_1 + \beta_2 x_2 + \varepsilon, \beta_1 x_1 + \beta_2 x_2 + \varepsilon) \\
&= \text{Cov}(\beta_1 x_1, \beta_1 x_1 + \beta_2 x_2 + \varepsilon) + \\
&\text{Cov}(\beta_2 x_2, \beta_1 x_1 + \beta_2 x_2 + \varepsilon) + \\
&\text{Cov}(\varepsilon, \beta_1 x_1 + \beta_2 x_2 + \varepsilon) \\
&= \text{Cov}(\beta_1 x_1, \beta_1 x_1) + \\
&\text{Cov}(\beta_2 x_2, \beta_1 x_1) + \\
&\text{Cov}(\varepsilon, \beta_1 x_1) + \\
&\text{Cov}(\beta_1 x_1, \beta_2 x_2) + \\
&\text{Cov}(\beta_2 x_2, \beta_2 x_2) + \\
&\text{Cov}(\varepsilon, \beta_2 x_2) + \\
&\text{Cov}(\beta_1 x_1, \varepsilon) + \\
&\text{Cov}(\beta_2 x_2, \varepsilon) + \\
&\text{Cov}(\varepsilon, \varepsilon) \\
&= \beta_1^2 \text{Cov}(x_1, x_1) + \\
&\beta_1 \beta_2 \text{Cov}(x_2, x_1) + \\
&0 \\
&\beta_1 \beta_2 \text{Cov}(x_1, x_2) + \\
&\beta_2^2 \text{Cov}(x_2, x_2) + \\
&0 + \\
&0 + \\
&0 + \\
&\text{Cov}(\varepsilon, \varepsilon) \\
&= \beta_1^2 \text{Var}(x_1) + \beta_2^2 \text{Var}(x_2) + 2\beta_1 \beta_2 \text{Cov}(x_1, x_2) + \text{Var}(\varepsilon)
\end{aligned}
$$

Finally, consider:



Exercise 5
Write down the structural equations for y_1 and y_2 and derive $\text{Var}(y_1)$.

Solution:

$$\begin{aligned}
\text{Var}(y_1) &= \text{Var}(\beta_1 x + \beta_3 y_2 + \varepsilon_1) \\
&= \text{Var}(\beta_1 x + \beta_3(\beta_2 x + \varepsilon_2) + \varepsilon_1) \\
&= \text{Var}(\beta_1 x + \beta_2 \beta_3 x + \beta_3 \varepsilon_2 + \varepsilon_1) \\
&= \text{Var}((\beta_1 + \beta_2 \beta_3) x + \beta_3 \varepsilon_2 + \varepsilon_1)
\end{aligned}$$

Given that we know residuals do not covary with anything:

$$\begin{aligned}
\text{Var}(y_1) &= \text{Var}((\beta_1 + \beta_2\beta_3)x) + \text{Var}(\beta_3\varepsilon_2) + \text{Var}(\varepsilon_1)) \\
&= (\beta_1 + \beta_2\beta_3)^2 \text{Var}(x) + \beta_3^2 \text{Var}(\varepsilon_2) + \text{Var}(\varepsilon_1)
\end{aligned}$$

Done. But we can expand:

$$\begin{aligned}
\text{Var}(y_1) = \beta_1^2 \text{Var}(x) + 2\beta_1\beta_2\beta_3 \text{Var}(x) + \beta_2^2\beta_3^2 \text{Var}(x) + \\
\beta_3^2 \text{Var}(\varepsilon_2) + \text{Var}(\varepsilon_1)
\end{aligned}$$