SEM 2: Structural Equation Modeling
Week 3 - Partial covariance

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If multivariate normality holds, then the Schur complement shows that any partial covariance can be expressed solely in terms of variances and covariances:

\[
\text{Cov}(y_i, y_j | x) = \text{Cov}(y_i, y_j) - \text{Cov}(y_i, x) \text{Var}(x)^{-1} \text{Cov}(x, y_j)
\]

in the multivariate case and:

\[
\text{Cov}(y_i, y_j | x) = \text{Cov}(y_i, y_j) - \frac{\text{Cov}(y_i, x) \text{Cov}(x, y_j)}{\text{Var}(x)}
\]

in the univariate case.
If multivariate normality holds, then the Schur complement shows that any partial covariance can be expressed solely in terms of variances and covariances:

\[
\text{Cov} (y_i, y_j | x) = \text{Cov} (y_i, y_j) - \text{Cov} (y_i, x) \text{Var} (x)^{-1} \text{Cov} (x, y_j)
\]

in the multivariate case and:

\[
\text{Cov} (y_i, y_j | x) = \text{Cov} (y_i, y_j) - \frac{\text{Cov} (y_i, x) \text{Cov} (x, y_j)}{\text{Var} (x)}
\]

in the univariate case.

- All conditional relationships can be expressed in terms of \( \Sigma \)!
- If we know \( \Sigma \), we know everything we can about the relationships between variables.
- Fitting a SEM model equals simultaneously testing all conditional independence relationships implied by the model!
Now using the Schur compliment:

\[
\text{Cov}(y_1, y_2 | \eta_1) = \text{Cov}(y_1, y_2) - \text{Cov}(y_1, \eta_1) \cdot \text{Var}(\eta_1) - \text{Cov}(\eta_1, y_2) = \psi_{11} \lambda_{21} - \psi_{11} \psi_{11} - 1 \lambda_{21} \psi_{11} = 0
\]

\[
\text{Cov}(y_1, y_2) = \psi_{11} \lambda_{21}
\]

\[
\text{Cov}(y_1, \eta_1) = \psi_{11}
\]

\[
\text{Cov}(y_2, \eta_1) = \lambda_{21} \psi_{11}
\]

\[
\text{Var}(\eta_1) = \psi_{11}
\]
Cov\( (y_1, y_2) = \psi_{11} \lambda_{21} \)
Cov\( (y_1, \eta_1) = \psi_{11} \)
Cov\( (y_2, \eta_1) = \lambda_{21} \psi_{11} \)
Var\( (\eta_1) = \psi_{11} \)

Now using the Schur compliment:
\[
\text{Cov} (y_1, y_2 \mid \eta_1) = \text{Cov} (y_1, y_2) - \\
\text{Cov} (y_1, \eta_1) \text{ Var} (\eta_1)^{-1} \text{Cov} (\eta_1, y_2) \]
\[
= \psi_{11} \lambda_{21} - \psi_{11} \psi_{11}^{-1} \lambda_{21} \psi_{11} \]
\[
= \psi_{11} \lambda_{21} - \psi_{11} \psi_{11}^{-1} \lambda_{21} \psi_{11} \]
\[
= \psi_{11} \lambda_{21} - \lambda_{21} \psi_{11} \]
\[
= 0 \]
\[
\begin{align*}
\text{Cov}(x, y_2) &= \beta_2 \beta_1 \text{Var}(x) \\
\text{Cov}(x, y_1) &= \beta_1 \text{Var}(x) \\
\text{Cov}(y_1, y_2) &= \beta_2 \beta_1^2 \text{Var}(x) + \beta_2 \theta_1 \\
\text{Var}(y_1) &= \beta_1^2 \text{Var}(x) + \theta_1
\end{align*}
\]
Cov(x, y_2) = \beta_2 \beta_1 \text{Var}(x) \\
Cov(x, y_1) = \beta_1 \text{Var}(x) \\
Cov(y_1, y_2) = \beta_2 \beta_1^2 \text{Var}(x) + \beta_2 \theta_1 \\
\text{Var}(y_1) = \beta_1^2 \text{Var}(x) + \theta_1

Now using the Schur compliment:

\begin{align*}
\text{Cov}(x, y_2 \mid y_1) &= \text{Cov}(x, y_2) - \\
&= \beta_2 \beta_1 \text{Var}(x) - \frac{\beta_1 \text{Var}(x)(\beta_2 \beta_1^2 \text{Var}(x) + \beta_2 \theta_1)}{\beta_1^2 \text{Var}(x) + \theta_1} \\
&= \beta_2 \beta_1 \text{Var}(x) - \beta_2 \beta_1 \text{Var}(x) \frac{\beta_1^2 \text{Var}(x) + \theta_1}{\beta_1^2 \text{Var}(x) + \theta_1} \\
&= \beta_2 \beta_1 \text{Var}(x) - \beta_2 \beta_1 \text{Var}(x) \\
&= 0
\end{align*}
\[ \text{Cov}(x_1, x_2) = 0 \]
\[ \text{Cov}(x_1, y) = \beta_1 \text{Var}(x_1) \]
\[ \text{Cov}(x_2, y) = \beta_2 \text{Var}(x_2) \]
\[ \text{Var}(y) = \theta + \beta_1^2 \text{Var}(x_1) + \beta_2^2 \text{Var}(x_2) \]
\[ \text{Cov}(x_1, x_2) = 0 \]
\[ \text{Cov}(x_1, y) = \beta_1 \text{Var}(x_1) \]
\[ \text{Cov}(x_2, y) = \beta_2 \text{Var}(x_2) \]
\[ \text{Var}(y) = \theta + \beta_1^2 \text{Var}(x_1) + \beta_2^2 \text{Var}(x_2) \]

Now using the Schur compliment:

\[ \text{Cov}(x_1, x_2 \mid y) = \text{Cov}(x_1, x_2) - \text{Cov}(x_1, y) \text{Var}(y)^{-1} \text{Cov}(x_2, y) \]
\[ = 0 - \frac{\beta_1 \text{Var}(x_1) \beta_2 \text{Var}(x_2)}{\theta + \beta_1^2 \text{Var}(x_1) + \beta_2^2 \text{Var}(x_2)} \]
\[ = \begin{cases} < 0 & \text{if sign}(\beta_1) = \text{sign}(\beta_2) \\ > 0 & \text{if sign}(\beta_1) \neq \text{sign}(\beta_2) \end{cases} \]
Psychological Networks in Clinical Populations: A tutorial on the consequences of Berkson's Bias

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ABSTRACT
In clinical research, populations are often selected on the sum-score of diagnostic criteria, i.e., symptoms. Estimating statistical models where a subset of the data is selected based on a function of the analyzed variables introduces Berkson's bias, which presents a potential threat to the validity of findings in the clinical literature. The aim ...