SEM 2: Structural Equation Modeling

Week 3 - Partial covariance

Sacha Epskamp

If multivariate normality holds, then the Schur complement shows that any partial covariance can be expressed solely in terms of variances and covariances:

$$\operatorname{Cov}(\boldsymbol{y}_i, \boldsymbol{y}_j \mid \boldsymbol{x}) = \operatorname{Cov}(\boldsymbol{y}_i, \boldsymbol{y}_j) - \operatorname{Cov}(\boldsymbol{y}_i, \boldsymbol{x}) \operatorname{Var}(\boldsymbol{x})^{-1} \operatorname{Cov}(\boldsymbol{x}, \boldsymbol{y}_j)$$

in the multivariate case and:

$$\operatorname{Cov}(y_i, y_j \mid x) = \operatorname{Cov}(y_i, y_j) - \frac{\operatorname{Cov}(y_i, x) \operatorname{Cov}(x, y_j)}{\operatorname{Var}(x)}$$

in the univariate case.

If multivariate normality holds, then the Schur complement shows that any partial covariance can be expressed solely in terms of variances and covariances:

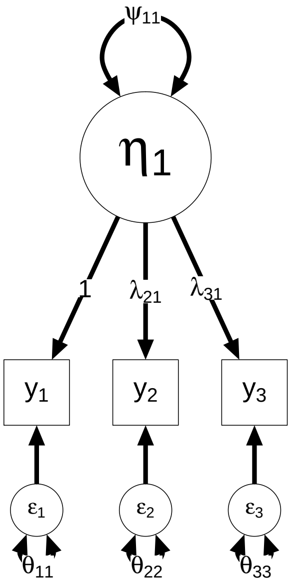
$$\mathrm{Cov}\left(\boldsymbol{y}_i, \boldsymbol{y}_j \mid \boldsymbol{x}\right) = \mathrm{Cov}\left(\boldsymbol{y}_i, \boldsymbol{y}_j\right) - \mathrm{Cov}\left(\boldsymbol{y}_i, \boldsymbol{x}\right) \mathrm{Var}\left(\boldsymbol{x}\right)^{-1} \mathrm{Cov}\left(\boldsymbol{x}, \boldsymbol{y}_j\right)$$

in the multivariate case and:

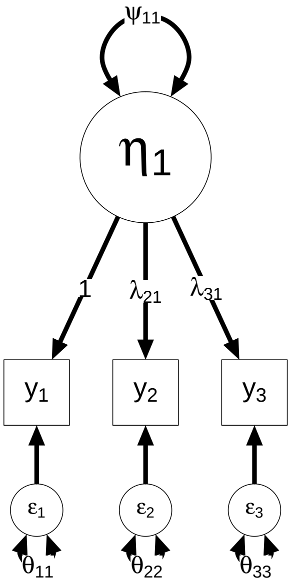
$$\mathrm{Cov}\left(y_i, y_j \mid x\right) = \mathrm{Cov}\left(y_i, y_j\right) - \frac{\mathrm{Cov}\left(y_i, x\right) \mathrm{Cov}\left(x, y_j\right)}{\mathrm{Var}\left(x\right)}$$

in the univariate case.

- All conditional relationships can be expressed in terms of $\boldsymbol{\Sigma}$!
- If we know $\boldsymbol{\Sigma}$, we know everything we can about the relationships between variables.
- Fitting a SEM model equals simultaneously testing all conditional independence relationships implied by the model!



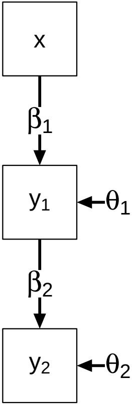
$$\mathrm{Cov}(y_1, y_2) = \psi_{11}\lambda_{21}$$
$$\mathrm{Cov}(y_1, \eta_1) = \psi_{11}$$
$$\mathrm{Cov}(y_2, \eta_1) = \lambda_{21}\psi_{11}$$
$$\mathrm{Var}(\eta_1) = \psi_{11}$$



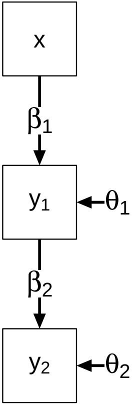
$$
\begin{aligned}
\mathrm{Cov}(y_1, y_2) &= \psi_{11}\lambda_{21} \\
\mathrm{Cov}(y_1, \eta_1) &= \psi_{11} \\
\mathrm{Cov}(y_2, \eta_1) &= \lambda_{21}\psi_{11} \\
\mathrm{Var}(\eta_1) &= \psi_{11}
\end{aligned}
$$

Now using the Schur compliment:

$$
\begin{aligned}
\mathrm{Cov}(y_1, y_2 \mid \eta_1) = \mathrm{Cov}\left(y_1, y_2\right) - \\
\mathrm{Cov}\left(y_1, \eta_1\right) \mathrm{Var}\left(\eta_1\right)^{-1} \mathrm{Cov}\left(\eta_1, y_2\right) \\
&= \psi_{11}\lambda_{21} - \psi_{11}\psi_{11}^{-1}\lambda_{21}\psi_{11} \\
&= \psi_{11}\lambda_{21} - \psi_{11}\psi_{11}^{-1}\lambda_{21}\psi_{11} \\
&= \psi_{11}\lambda_{21} - \lambda_{21}\psi_{11} \\
&= 0
\end{aligned}
$$



$$\mathrm{Cov}(x, y_2) = \beta_2\beta_1\mathrm{Var}(x)$$
$$\mathrm{Cov}(x, y_1) = \beta_1\mathrm{Var}(x)$$
$$\mathrm{Cov}(y_1, y_2) = \beta_2\beta_1^2\mathrm{Var}(x) + \beta_2\theta_1$$
$$\mathrm{Var}(y_1) = \beta_1^2\mathrm{Var}(x) + \theta_1$$



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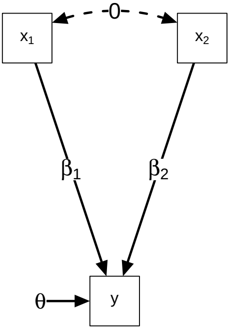
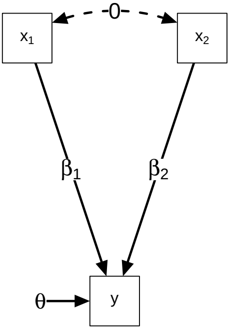
$$\mathrm{Var}(y_1) = \beta_1^2 \mathrm{Var}(x) + \theta_1$$

Now using the Schur compliment:

$$\begin{aligned}
\mathrm{Cov}(x, y_2 \mid y_1) &= \mathrm{Cov}\left(x, y_2\right) - \\
&\quad \mathrm{Cov}\left(x, y_1\right) \mathrm{Var}\left(y_1\right)^{-1} \mathrm{Cov}\left(y_2, y_1\right) \\
&= \beta_2 \beta_1 \mathrm{Var}(x) - \frac{\beta_1 \mathrm{Var}(x)(\beta_2 \beta_1^2 \mathrm{Var}(x) + \beta_2 \theta_1)}{\beta_1^2 \mathrm{Var}(x) + \theta_1} \\
&= \beta_2 \beta_1 \mathrm{Var}(x) - \beta_2 \beta_1 \mathrm{Var}(x) \frac{\beta_1^2 \mathrm{Var}(x) + \theta_1}{\beta_1^2 \mathrm{Var}(x) + \theta_1} \\
&= \beta_2 \beta_1 \mathrm{Var}(x) - \beta_2 \beta_1 \mathrm{Var}(x) \\
&= 0
\end{aligned}$$



$$\mathrm{Cov}(x_1, x_2) = 0$$
$$\mathrm{Cov}(x_1, y) = \beta_1 \mathrm{Var}(x_1)$$
$$\mathrm{Cov}(x_2, y) = \beta_2 \mathrm{Var}(x_2)$$
$$\mathrm{Var}(y) = \theta + \beta_1^2 \mathrm{Var}(x_1) + \beta_2^2 \mathrm{Var}(x_2)$$



$$\mathrm{Cov}(x_1, x_2) = 0$$
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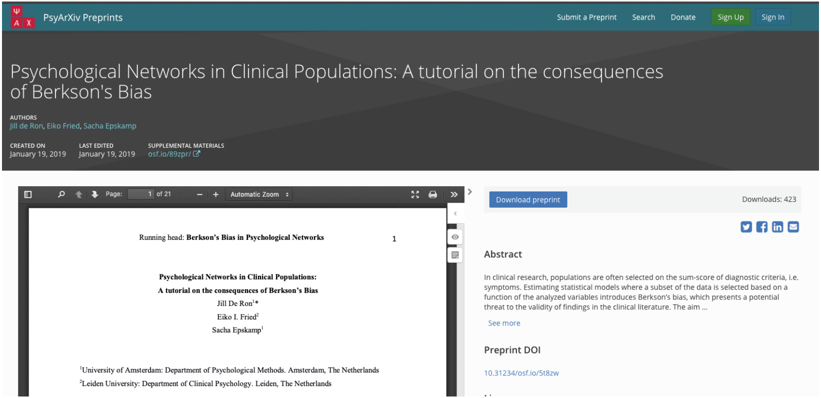
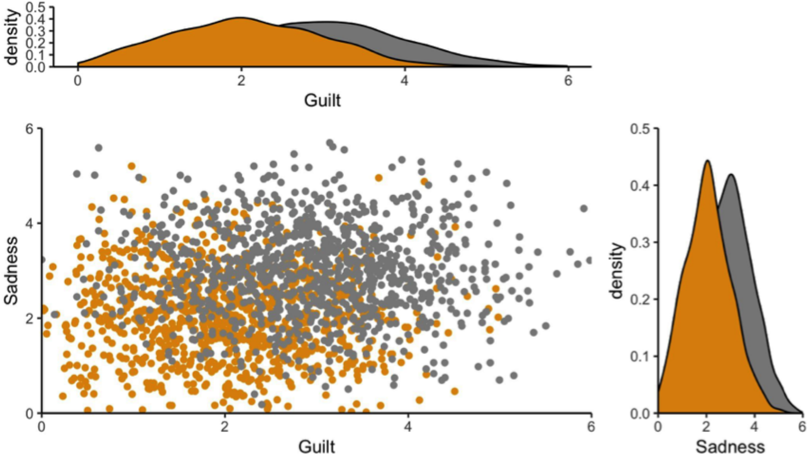
$$\mathrm{Cov}(x_2, y) = \beta_2 \mathrm{Var}(x_2)$$
$$\mathrm{Var}(y) = \theta + \beta_1^2 \mathrm{Var}(x_1) + \beta_2^2 \mathrm{Var}(x_2)$$

Now using the Schur compliment:

$$\begin{aligned}\mathrm{Cov}(x_1, x_2 \mid y) &= \mathrm{Cov}\left(x_1, x_2\right) - \mathrm{Cov}\left(x_1, y\right) \mathrm{Var}\left(y\right)^{-1} \mathrm{Cov}\left(x_2, y\right) \\ &= 0 - \frac{\beta_1 \mathrm{Var}(x_1) \beta_2 \mathrm{Var}(x_2)}{\theta + \beta_1^2 \mathrm{Var}(x_1) + \beta_2^2 \mathrm{Var}(x_2)} \\ &= \begin{cases} {\color{red}< 0} & \text{if } \mathrm{sign}(\beta_1) = \mathrm{sign}(\beta_2) \\ {\color{blue}> 0} & \text{if } \mathrm{sign}(\beta_1) \neq \mathrm{sign}(\beta_2) \end{cases}\end{aligned}$$



https://psyarxiv.com/5t8zw/



density
0.5
0.4
0.3
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0.1
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Guilt
Sadness
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2
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0
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Guilt
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density
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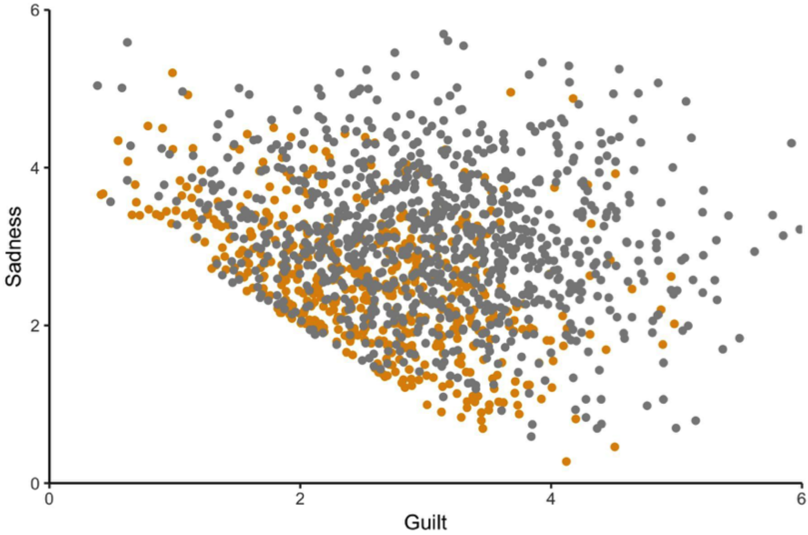

Sadness
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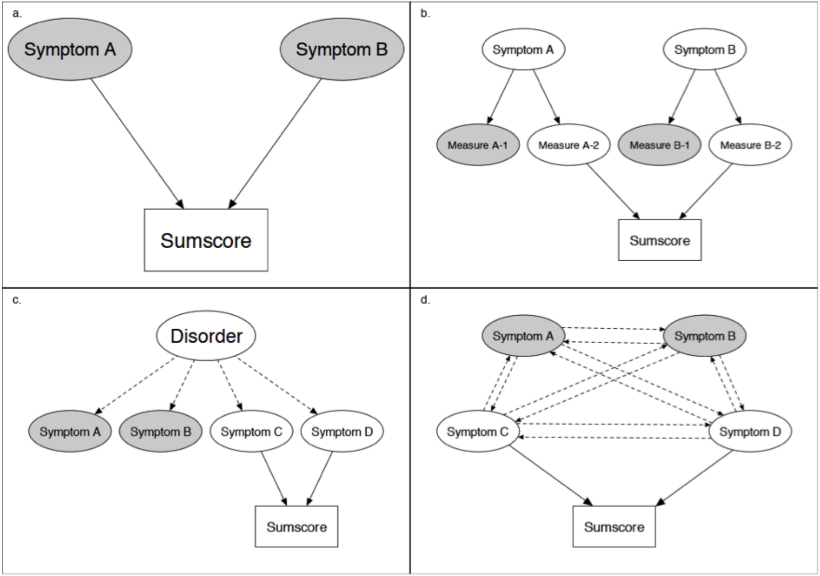
a.
Symptom A
Symptom B
Sumscore
b.
Symptom A
Symptom B
Measure A-1
Measure A-2
Measure B-1
Measure B-2
Sumscore
c.
Disorder
Symptom A
Symptom B
Symptom C
Symptom D
Sumscore
d.
Symptom A
Symptom B
Symptom C
Symptom D
Sumscore